PERGAMON

International Journal of Solids and Structures 40 (2003) 807–825

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A note on the crack analogue model for fretting fatigue

M. Ciavarella [a,b,*], G. Macina [b]

[a] CNR-ITC, Str. Crocefisso 2/B, 70126 Bari, Italy
[b] CEMEC—Politecnico di Bari, via Amendola 126/B, 70126 Bari, Italy

Received 31 May 2002; received in revised form 29 October 2002

Abstract

The contact of a flat punch over a half-plane under constant normal loads, and oscillating tangential and bulk loads is studied, with the aim to improve the crack analogue (CA) model for fretting fatigue (FF) (Acta Mater. 46(9) (1998) 2955). New analytical results are found for a range of conditions, finding the effect of bulk loads and of partial slip which were not considered in the original CA model. Implications for the FF life assessment methodology are found to be significant.

Keywords: Crack; Fatigue; Loads

1. Introduction

Fretting fatigue (FF) has been discovered long time ago (Tomlinson, 1927; Eden et al., 1911), but has mostly been seen, until recently, as a "separate" area of fatigue, where the mechanical damage over the surface was considered to have a dominant role in decreasing the fatigue performance of the material. Therefore, parameters as microslip amplitude and surface energy dissipated by friction were considered: typically, the effect of fretting was measured on simple rigs applied to the standard fatigue machines, and fretting was seen as an additional effect over an otherwise undisturbed standard rotating bending or push–pull fatigue test. Then, the FF test was compared and empirically correlated with unfretted specimen SN curves (Nishioka and Hirakawa, 1969), like

$$\sigma_{\mathrm{FFL}} = \sigma_{\mathrm{FL}} - fp_m\left(1 - \mathrm{e}^{(-\delta/k)}\right) \tag{1}$$

where σ_{FFL} is the "fretting fatigue limit", σ_{FL} is the standard fatigue limit, f is friction coefficient, p_m is the applied pressure, δ is the amplitude of microslip and k is an empirical constant depending on materials and surface conditions. Other approaches considered hybrid damage parameters also including the local surface stress (Nowell and Hills, 1990), like the Ruiz' parameter $R = \sigma\tau\delta < (\sigma\tau\delta)_{\mathrm{crit}}$. The existence of a FF material

* Corresponding author. Address: CNR IRIS, COMES Computational Mechanics of Solids, Str. Crocefisso 2/B, 70126 Bari, Italy. Fax: +39-080-5482533.
E-mail address: m.ciavarella@area.ba.cnr.it (M. Ciavarella).

0020-7683/02/$ - see front matter
PII: S0020-7683(02)00652-2

(or contact pair) property such as R_{crit} was never satisfactorily proved, and its determination remained empirical and unrelated to more classical fatigue literature.

More recently, the role of the contact stress field in provoquing *fatigue from a stress raiser feature*, has been recognized, and indeed a *crack analogue* (CA) model for the case where the contact is complete (singular pressure and frictional shear tractions) and a *notch analogue* (NA) for cases where we expect a smooth transition to zero pressure at the contact area edges, and correspondingly a finite stress concentration, have been proposed in two remarkable papers by the MIT group, called in the following CA and NA models (Giannakopoulos et al. (1998, 2000), respectively for CA and NA models). It was recognized that the stress field induced by the contact is very similar to the square-root singular stress field around an external crack—the singular stress field can be quantified by a stress intensity factor, and the bulk stress in the contacting materials becomes a T-stress in the fracture mechanics terminology. Cracks developed at the contact site are kinked cracks, and the condition of initiation is rather a condition for non-propagation over stress intensity factors ranges $\Delta K < \Delta K_{th}$. Therefore, for the little crack of size $l \ll 2a$ (see Fig. 1a) where $2a$ is the size of the main crack, no correction should be used according to short-crack theories, as the fracture mechanics conditions (both threshold and propagation regimes) can be written directly in terms of the main

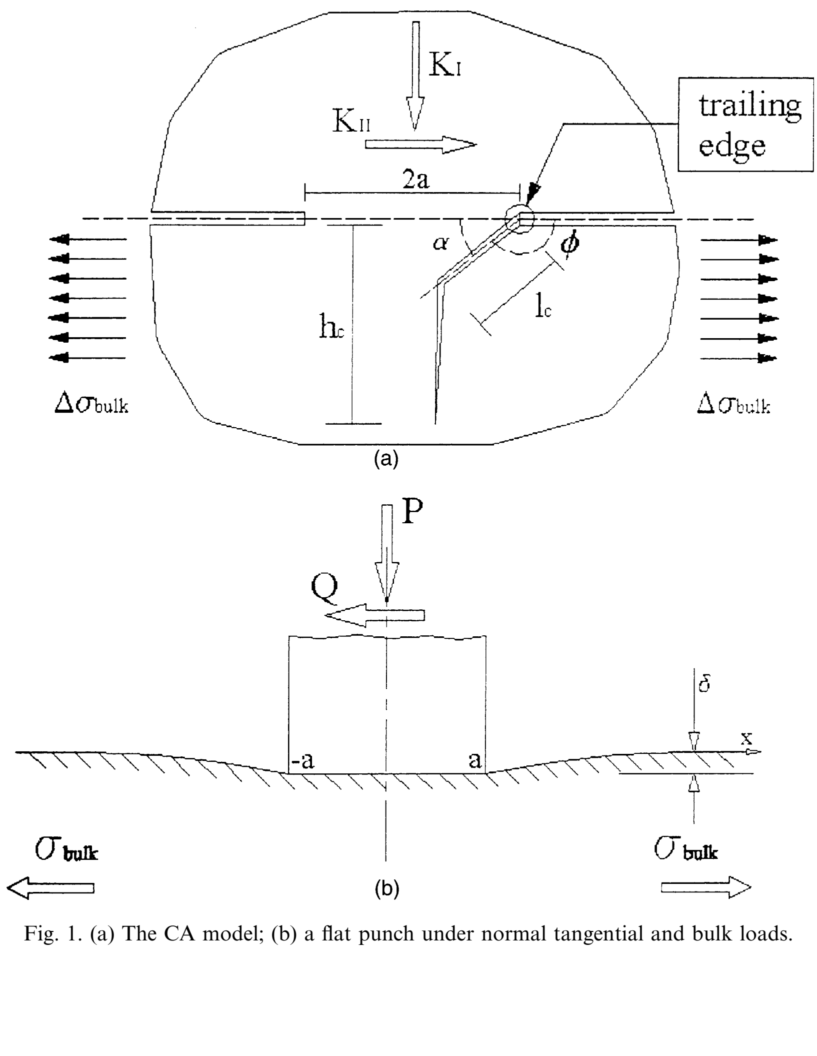

Fig. 1. (a) The CA model; (b) a flat punch under normal tangential and bulk loads.

crack and it is the total size of the crack which matters in determining conditions for the necessity of short or small crack corrections.

In the original CA model, a simplified analysis is made, neglecting the effect of bulk stress into the tractions, and a full stick condition was assumed. However, in general, frictional problems are known to cause partial slip, i.e. regions of stick and microslip both simultaneously present within the contact area. In the classical case of Hertzian contact, when tangential load is applied sequentially to a (constant) normal load Cattaneo (1938), and later independently Mindlin (1949) found the solution where an annulus of slip arises of increasing dimension for increasing tangential load, up to the condition of full sliding, reached when $Q = fP$. More details on the classical theory and experimental evidence for partial slip are well detailed in Sections 7.2–7.4 of Johnson's book (1985).

A more general solution to the Cattaneo–Mindlin problem for any plane contact problem (not necessarily Hertzian) was given in (Ciavarella, 1998a,b), and Jaeger (1998), finding that the Cattaneo idea of correcting the full sliding shear traction term $q(x) = fp(x)$ could be generalized, where f is friction coefficient and $p(x)$ the normal contact pressure distribution. Specifically, the corrective shear stress $q^*(x)$ term

$$q^*(x) = p(x) - q(x)/f \tag{2}$$

is shown to correspond to the normal contact pressure distribution at some smaller value of the normal load. Therefore, the increase of tangential force simply make the stick zone to shrink in the reverse order as the normal contact area during the normal loading process. [1]

The effect of bulk stress, oscillating in phase with the tangential loading, was first considered for the plane Hertzian geometry by Nowell and Hills (1987), and later generalized to more general geometries in (Ciavarella et al., 1999). In both cases, surface strain e_x is *moderate* in order to assume that slip zones of the same sign occur at both ends of contact, and a solution similar to (2) is found, where the corrective pressure, $q^*(x)$, is found from the original equation for normal contact, for a lower normal load and an additional fictitious *geometrical* rotation e_x/f. The condition for this 'corrective' solution to hold is that the stick zone has to be contained entirely within the contact area. Here we will consider both the case of *moderate* bulk stress, and of large bulk stress, and indeed the limit case of bulk stress only (i.e. no tangential load) obtaining analytical solutions. In the original CA model, only the effect of tangential load on ΔK_{II} was considered, and the only possible conditions was full stick (or, in the limit, full slip). In the present paper, bulk strain is found to significantly alter the tractions (and the stress intensity factors regulating the FF life), and possibly induce partial slip.

It should be noticed that, as in the original CA model, only the case of constant normal load P is here considered. Although in most real application, for example in a dovetail joint of a typical gas turbine engine, there are various components of load that are varying with time, many fretting tests (validating methodologies and materials) are conducted in the case of constant normal load, and this is a sufficient motivation for the detailed analysis of the present paper.

2. The crack analogue model

We consider the model suggested by Giannakopoulos et al. (1998), i.e. a square-ended foot pressing over a fatigue specimen, according to Fig. 1. For a constant mode I load and a varying mode II load (constant normal load P and varying Q), it is found (for a 2D geometry)

[1] A further corollary (Ciavarella and Hills, 1999), is that any wear altering the geometry in the region of the microslip, will not change the dimension and location of the region of stick, where the pressure will progressively localize, and in the limit a singularity of pressure would arise at the stick–slip transition point.

$$K_{\mathrm{I}} = -\frac{P}{\sqrt{\pi a}} = -\frac{2}{\pi}\bar{p}\sqrt{\pi a}$$
$$\Delta K_{\mathrm{II}} = \pm\frac{Q}{\sqrt{\pi a}} = \pm\frac{2}{\pi}\bar{q}\sqrt{\pi a} \tag{3}$$

where $\bar{p} = P/2a$ and $\bar{q} = Q/2a$. The corresponding contact tractions are

$$\sigma_{yy} = -\frac{P}{\pi\sqrt{a^2 - x^2}} \tag{4}$$

$$\sigma_{xy} = \frac{Q}{\pi\sqrt{a^2 - x^2}} \tag{5}$$

This solution holds up to the limiting case of full sliding, inclusive, as in that case simply $Q = fP$, and $\bar{q} = f\bar{p}$. However, this only holds true when no bulk stress is present which is not the typical case and not the critical case either, because it is the combination of bulk fatigue and fretting loads which causes the most serious problems. An additional, but minor, assumption, is to neglect the effect of material dissimilarity, which would couple the pressure and shear terms. We remind that elastically similar does not mean necessarily identical materials, but more generally Dundurs' second constant β to be zero (Hills et al., 1993), i.e.

$$\beta = \frac{\frac{\mu_2}{\mu_1}(\kappa_1 - 1) - (\kappa_2 - 1)}{\frac{\mu_2}{\mu_1}(\kappa_1 + 1) + (\kappa_2 + 1)} = 0 \tag{6}$$

Here, κ_i is the Kolosov constant, equal to $(3 - \nu)/(\nu + 1)$ under plane stress, and $(3 - 4\nu)$ under plane strain conditions, where ν_i, μ_i are Poisson's ratio and shear modulus of body i. Therefore, the condition $\beta = 0$ can be written in plane strain, for example, as

$$\frac{\mu_2}{\mu_1} = \frac{1 - 2\nu_2}{1 - 2\nu_1} \tag{7}$$

In the original CA model paper some reference is made to the effect of elastic dissimilarity in the simplest case of pure normal load of a rigid punch on a compressible half-plane (Eqs. (14) and (15) of Giannakopoulos et al., 1998). This leads, in the case of perfect stick, to non-standard singularities (such as the logarithmic one), and more generally a wide range of singularities are obtained for general elastic wedge in frictional contact over a half-plane (Mugadu, 2002). In order to consider the simplest case of no-coupling, $\beta = 0$, and yet consider the half-plane elasticity correct, the simple solution above (and the entire treatment which follows) is rigorous only if the indenting punch is rigid $\mu_1 \to \infty$, and the indented material incompressible $\nu_2 = 1/2$ (so that both sides of Eq. (7) tend to zero).

Turning back on the original CA model, the basic idea is to consider the contact area as equivalent to a long-crack of dimension a, loaded by K_{I}, ΔK_{II}, with the following steps, concerning: (1) initiation condition; (2) direction of propagation; (3) condition for self-arrest:

(1) For the initiation condition, a threshold condition is written in terms of the "equivalent crack" according to LEFM as

$$\Delta K_{\mathrm{II}} = \pm\frac{Q}{\sqrt{\pi a}} = \pm\frac{2}{\pi}\bar{q}\sqrt{\pi a} \leqslant \Delta K_{\mathrm{II,th}} \tag{8}$$

where generally it is sufficient (and conservative) to assume $\Delta K_{\mathrm{II,th}} = \Delta K_{\mathrm{th}}$ where ΔK_{th} is the mode I long-crack fatigue threshold. Also, consistency requires that ΔK_{th} and ΔK_{II} are range values for the given load ratio R. No indication is given for the contribution of K_{I} or the bulk stress. The latter has two effects: (i) to modify shear tractions and accordingly the mode II stress intensities ΔK_{II}; (ii) to act as an independent bulk fatigue loading when it is cycling. In the original CA model, the bulk stress is suggested to act as a T-stress,

$$T = \sigma_{xx} = \sigma_{\mathrm{b}} \tag{9}$$

but this is not further taken into account (as there is no simple method to include the effect of T-stress for fatigue threshold calculations). Notice therefore that in the original CA model the effect of bulk stress is completely neglected except in the following third step;

(2) If initiated, a kinked crack would experience the following SIFs, depending on the main SIFs K_{I}, K_{II}:

$$\begin{aligned} k_{\mathrm{I}} &= a_{11}(\phi)K_{\mathrm{I}} + a_{12}(\phi)K_{\mathrm{II}} \\ k_{\mathrm{II}} &= a_{21}(\phi)K_{\mathrm{I}} + a_{22}(\phi)K_{\mathrm{II}} \end{aligned} \tag{10}$$

where $a_{11}(\phi), \ldots, a_{22}(\phi)$ are the classical Cotterel and Rice geometric functions for a kinked infinitesimal crack in 2D

$$\begin{aligned} a_{11}(\phi) &= \frac{1}{4}\left(3\cos\frac{\phi}{2} + \cos\frac{3\phi}{2}\right) \\ a_{12}(\phi) &= -\frac{3}{4}\left(\sin\frac{\phi}{2} + \sin\frac{3\phi}{2}\right) \\ a_{21}(\phi) &= \frac{1}{4}\left(\sin\frac{\phi}{2} + \sin\frac{3\phi}{2}\right) \\ a_{22}(\phi) &= \frac{1}{4}\left(\cos\frac{\phi}{2} + 3\cos\frac{3\phi}{2}\right) \end{aligned} \tag{11}$$

The mode I of the main crack, K_{I}, gives no contribution because it is not oscillating (although the kinked crack experiences both a constant mode I and a constant mode II because of the K_{I}), i.e.

$$\begin{aligned} \Delta k_{\mathrm{I}} &= \Delta k_{\mathrm{I}}(\Delta K_{\mathrm{II}}) \\ \Delta k_{\mathrm{II}} &= \Delta k_{\mathrm{II}}(\Delta K_{\mathrm{II}}) \end{aligned} \tag{12}$$

although the actual minimum, mean and maximum values of k_{I}, k_{II}, do depend on K_{I}, too. Then, the original CA model suggests that the *initiation direction* will be the one for which $k_{\mathrm{II}} = 0$ (at the max value of load) which translates into

$$a_{21}(\phi)/a_{22}(\phi) = -K_{\mathrm{II}}/K_{\mathrm{I}} = \overline{q}/\overline{p} \tag{13}$$

which implicitly is imposing the condition only at the maximum value of K_{II}, given that K_{I} is in reality constant, and the condition $k_{\mathrm{II}} = 0$ is in reality matched at other angles for intermediate values of the load during the cycle. The simplified condition above gives

$$\frac{\sin\frac{\phi}{2} + \sin\frac{3\phi}{2}}{\cos\frac{\phi}{2} + 3\cos\frac{3\phi}{2}} = -K_{\mathrm{II}}/K_{\mathrm{I}} = \overline{q}/\overline{p} \leqslant f \tag{14}$$

where f is the local friction coefficient. The larger is the ratio $\overline{q}/\overline{p} < f$, the larger is ϕ. For example, for $\overline{q}/\overline{p} = 0.5$, we get $\phi = 40°$.

(3) If no bulk stress were present, the crack even if initiated would not propagate too far into the specimen. Accordingly, a second condition of interest is the following; the crack, once initiated, will kink at a distance l_c from the surface. If at this distance, the bulk stress is not high enough, the crack will arrest.

Therefore, the CA computes the crack driving force at this distance and compares it again with the long-crack threshold. This condition is written in terms of the bulk stress alone in the form

$$\Delta K_{\mathrm{th}} = \frac{\Delta\sigma_{\mathrm{b}}}{4}\sqrt{\pi l_c}F(\phi) \tag{15}$$

where $F(\phi)$ is a calibrating function for the kinked crack. Without going into the details of the derivation of the latter equation (as it will not be further pursued here), notice that the driving force here is the bulk stress *alone* as the condition for crack arrest of a kinked crack if initiated. As the contact stress field itself may give additional driving force to the crack, it may be under-conservative to neglect it.

The present paper improves some aspects of the contact mechanics in the presence of bulk stress, and gives consequent modifications of the CA model. In particular, the condition of initiation (or non-propagation of the existing "crack equivalent" contact) is considered in detail, and the bulk stress is included in the calculation of the ΔK factors. Accordingly, the direction of propagation is modified. Since there is no special reason to consider separately the effect of contact loads and bulk stress (neglecting the former, when considering the latter, and vice versa), the effect of contact loads and of bulk stress are considered at the same time, writing a single condition for initiation, giving implicitly the condition of self-arrest.

More improvements of the CA model in order to consider varying contact loads, or effect of different R-ratios, are not considered here.

3. Contact problem

We turn back to the contact problem in Fig. 1, which is governed by standard integral equations given in many books on contact mechanics (see e.g. Johnson, 1985; Hills et al., 1993), obtained simply by integrating the Flamant solution for a line force acting on the half-plane. In particular, returning to the case of two elastic materials, we write a first equation in terms of the tangential displacements derivative $\partial u_{x1}/\partial x$ for the punch

$$\frac{\partial u_{x1}}{\partial x}(x,0) = \frac{\kappa_1 - 1}{4\mu_1}p(x) - \frac{\kappa_1 + 1}{4\pi\mu_1}\int_{-a}^{a}\frac{q(t)}{x-t}\mathrm{d}t \tag{16}$$

and similarly for the half-plane, including a independent bulk strain as external loading condition,

$$\frac{\partial u_{x2}}{\partial x}(x,0) = \frac{\kappa_2 - 1}{4\mu_2}p(x) + \frac{\kappa_2 + 1}{4\pi\mu_2}\int_{-a}^{a}\frac{q(t)}{x-t}\mathrm{d}t + e_{\mathrm{b}} \tag{17}$$

The bulk strain e_{b} is obviously related to the bulk stress in material labelled as "2", for plain strain as

$$e_{\mathrm{b}} = \frac{\sigma_{\mathrm{b}}}{E_2}(1-\nu_2^2) \tag{18}$$

Coulomb's friction law implies conditions on the relative tangential displacements function

$$g'(x) = \frac{\partial u_{x1}}{\partial x} - \frac{\partial u_{x2}}{\partial x} = -\frac{A}{\pi}\int_{-a}^{a}\frac{q(t)\,\mathrm{d}t}{x-t} + \beta A p(x) - \frac{\sigma_{\mathrm{b}}}{E_2}(1-\nu_2^2) \tag{19}$$

where A is the "composite compliance" of the bodies which in terms of Young's modulus and Poisson's ratio of each material, is

$$A = \frac{2(1-\nu_1^2)}{E_1} + \frac{2(1-\nu_2^2)}{E_2} = 2\left(\frac{1}{E_1^*} + \frac{1}{E_2^*}\right) \tag{20}$$

We recollect here that we have assumed $\beta = 0$, and by writing [2]

$$\left(\frac{E_2^*}{E_1^*} + 1\right) = \gamma \tag{21}$$

we can finally write the compatibility condition on displacements as

$$\frac{g'(x)}{A} = -\frac{1}{\pi}\int_{-a}^{a} \frac{q(t)}{x-t}\,\mathrm{d}t - \frac{\sigma_b}{2\gamma} \tag{22}$$

In the stick region, assuming there is no previous slip (as it is correct after the normal load has been applied and no tangential or bulk load), we can write $g'(x) = 0$

$$\int_{-a}^{a} \frac{q(t)}{x-t}\,\mathrm{d}t = -\frac{\pi\sigma_b}{2\gamma}, \quad x \in \text{stick zone} \tag{23}$$

The solution is subject to Coulomb's law for friction, and in particular $|q(x)| \leqslant fp(x)$ with the additional constraint that frictional slip opposes relative motion; finally, equilibrium in the horizontal direction gives

$$Q = \int_{-a}^{a} q(x)\,\mathrm{d}x$$

where obviously Q is the applied tangential load.

Once shear tractions are determined, the resulting surface stress in the half-plane can be obtained from (17) and (18). In fact,

$$e_{xx}(x,0) = \frac{\partial u_{x2}}{\partial x} = \frac{(1+\nu_2)(1-2\nu_2)}{E_2}p(x) + \frac{2(1-\nu_2^2)}{\pi E_2}\int_{-a}^{a}\frac{q(t)}{x-t}\,\mathrm{d}t + \frac{\sigma_b}{E_2}(1-\nu_2^2) \tag{24}$$

Also, from the constitutive equations under plane strain,

$$e_{xx} = \frac{(1-\nu^2)\sigma_{xx}}{E} - \frac{\nu(1+\nu)\sigma_{yy}}{E} \tag{25}$$

and considering $\sigma_{yy}(x,0) = p(x)$, we get

$$\sigma_{xx}(x,0) = \frac{E_2}{(1-\nu_2^2)}e_{xx}(x,0) + \frac{\nu_2}{(1-\nu_2)}p(x) \tag{26}$$

and therefore, by combining (24)–(26)

$$\sigma_{xx}(x,0) = p(x) + \frac{2}{\pi}\int_{-a}^{a}\frac{q(t)}{x-t}\,\mathrm{d}t + \sigma_b \tag{27}$$

which we will be using to obtain details on surface stress in the various cases—notice that here γ has no effect, whereas it has effect on the determination of the determination of the stick region, as in Eq. (23).

[2] Notice that $\gamma = 2$ for identical materials, but is equal to 1 for the case most rigorously treated in the present paper, of rigid punch and incompressible material—we continue to write γ in the following in order to obtain the solution in general, although approximate for $\gamma \neq 1$.

3.1. Tangential load only

The shear tractions are given by the solution of the integral equation above (23), where $g'(x)$ depends on relative displacements in tangential direction, in the following manner: when a tangential load Q is applied sequentially to the normal load only, obviously $g'(x) = 0$ in the stick region, as already used in deriving (23). The equation is the same found for the normal load only, so the solution is as correctly derived in the original CA model. This solution in fact satisfies Coulomb's law for friction, and in particular $|q(x)| \leqslant f|p(x)|$ in the entire contact area, as long as $|Q| \leqslant f|P|$. At this limit, the is suddenly full sliding in the entire contact area.

After determining the contact tractions, it is possible to quantify the tensile stress field generated. In particular, the surface stress, is obtained as

$$\sigma_{xx}(x,0) = \frac{4Q}{\pi^2\sqrt{x^2-a^2}}\left(\arctan\sqrt{\frac{x-a}{x+a}} + \arctan\sqrt{\frac{x+a}{x-a}}\right), \quad |x| \geqslant \sigma_{xx}(x,0)a = p(x), \quad |x| \leqslant a \tag{28}$$

and this is plotted in Fig. 2 where it appears clearly that the only tensile stresses are at the trailing edge of the contact. Near the contact edge, the asymptotic form for σ_{xx} is, for example for the right edge (and symmetrically for the left edge)

$$\begin{aligned} \sigma_{xx}(x \to a^+) &= \lim_{x\to a^+} \sigma_{xx}(x,0) = 2\frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \\ \sigma_{xx}(x \to a^-) &= \lim_{x\to a^-} \sigma_{xx}(x,0) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \end{aligned} \tag{29}$$

as predicted by the CA model, where K_{I} and K_{II} are defined in (3).

3.2. Bulk stress only

In the case of normal force P and bulk load σ_b the solving Eq. (23) in the hypothesis of full stick gives

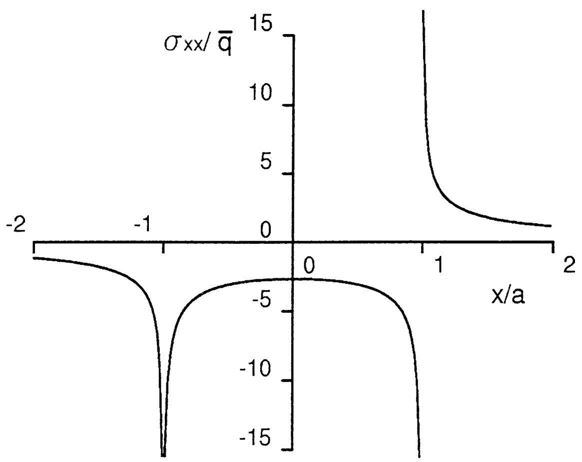

Fig. 2. Tangential load only: surface stress $\sigma_{xx}/\bar{q}$ for $Q/fP = 0.5$.

$$q(x) = \frac{\sigma_b/2\gamma}{\pi\sqrt{a^2 - x^2}} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}}{x - t} dt = \frac{x/a}{\sqrt{1 - (x/a)^2}} \frac{\sigma_b}{2\gamma} \tag{30}$$

which holds if $|q(x)| \leqslant fp(x)$, i.e. if

$$\frac{|x|/a}{\sqrt{1 - (x/a)^2}} \frac{\sigma_b}{2\gamma} \leqslant \frac{2}{\pi} \frac{f\bar{p}}{\sqrt{1 - (x/a)^2}} \tag{31}$$

Hence, there is complete stick only for small bulk loads, when

$$\sigma_b \leqslant \frac{4}{\pi} \gamma f \bar{p} \tag{32}$$

For larger bulk loads, if $\sigma_b \geqslant (4/\pi)\gamma f\bar{p}$, two slip zone take place next to the contact edge in symmetrical position, and the solution of the integral equation is obtained using a procedure similar to Spence's solution (Spence, 1973). The solution of tangential traction can be considered as

$$q(x) = \begin{cases} q^*(x), & |x| \leqslant b \\ fp(x)\,\mathrm{sign}(x), & b \leqslant |x| \leqslant a \end{cases} \tag{33}$$

where b is the semidimension of the stick area.

Hence, from the solving Eq. (23) we can write

$$-\int_{-a}^{-b} \frac{fp(t)}{x - t} dt + \int_{-a}^{a} \frac{q^*(t)}{x - t} dt + \int_{b}^{a} \frac{fp(t)}{x - t} dt + \frac{\sigma_b \pi}{2\gamma} = 0, \quad |x| \leqslant b \tag{34}$$

Considering the skew-symmetrical properties of $p(t)$ and $q(t)$,

$$\int_{-a}^{a} \frac{q^*(t)}{x - t} dt = 2 \int_{0}^{b} \frac{q^*(t)}{x^2 - t^2} t dt \tag{35}$$

and

$$\begin{aligned} -\int_{-a}^{-b} \frac{fp(t)}{x - t} dt + \int_{b}^{a} \frac{fp(t)}{x - t} dt &= -\int_{-a}^{-b} \frac{fp(t)}{x^2 - t^2}(x + t) dt + \int_{b}^{a} \frac{fp(t)}{x^2 - t^2}(x + t) dt \\ &= 2 \int_{b}^{a} \frac{fp(t)}{x^2 - t^2} t\, dt \end{aligned} \tag{36}$$

Finally, we get a Cauchy integral equation of the second kind (Hills et al., 1993)

$$\frac{2}{\pi} \int_{0}^{b} \frac{q^*(t)}{x^2 - t^2} t\, dt = -\frac{\sigma_b}{2\gamma} - \frac{4}{\pi^2} \int_{b}^{a} \frac{f\bar{p}}{(x^2 - t^2)\sqrt{1 - (t/a)^2}} t\, dt = F(x), \quad |x| \leqslant b \tag{37}$$

whose solution is

$$q^*(x) = -\frac{2}{\pi} x\sqrt{b^2 - x^2} \int_{0}^{b} \frac{F(y)}{\sqrt{b^2 - y^2}(x^2 - y^2)} dy \tag{38}$$

This solution is only valid if the following consistency condition is satisfied

$$\int_{0}^{b} \frac{F(y)}{\sqrt{b^2 - y^2}} dy = 0 \tag{39}$$

From (37) the condition above becomes

$$-\int_{0}^{b} \frac{\sigma_b/2\gamma}{\sqrt{b^2 - y^2}} dy + \frac{4}{\pi^2} \int_{0}^{b} \frac{\int_{b}^{a} \frac{f\bar{p}}{(t^2 - y^2)\sqrt{1 - (t/a)^2}} t\, dt}{\sqrt{b^2 - y^2}} dy = 0 \tag{40}$$

and in particular the first term is equal to $\pi\sigma/4\gamma$, whereas the second one can be suitably write as

$$\frac{4}{\pi^2}\int_b^a \frac{f\bar{p}}{\sqrt{1-(t/a)^2}}t\left[\int_0^b \frac{1}{(t^2-y^2)\sqrt{b^2-y^2}}\mathrm{d}y\right]\mathrm{d}t \tag{41}$$

where the integral in square brackets is equal to

$$\frac{\pi/2}{t\sqrt{t^2-b^2}} \tag{42}$$

Hence, we have

$$-\frac{\pi\sigma_\mathrm{b}}{4\gamma}+\frac{2}{\pi}\int_{b/a}^1 \frac{f\bar{p}}{\sqrt{1-t^2}\sqrt{t^2-(b/a)^2}}\mathrm{d}t=0 \tag{43}$$

Imposing the substitution $z=\sqrt{1-t^2}$ we find for the integral above

$$\int_0^{\sqrt{1-(b/a)^2}} \frac{f\bar{p}}{\sqrt{1-z^2}\sqrt{1-(b/a)^2-z^2}}\mathrm{d}z=f\bar{p}K'(b/a) \tag{44}$$

Hence, we get

$$K'(b/a)=\frac{\pi^2\sigma_\mathrm{b}}{8\gamma f\bar{p}} \tag{45}$$

where $K'(b/a)=K(1-(b/a)^2)$, and $K(*)$ is the complete elliptic integral of the second kind.

In Fig. 3 the stick area semiwidth is plotted as a function of dimensionless bulk load $\sigma_\mathrm{b}/(4/\pi)\gamma f\bar{p}$ and it can be noticed that for $\sigma_\mathrm{b}/(4/\pi)\gamma f\bar{p}\leqslant 1$ the entire contact area is in full stick.

Using the same method we can simplify the expression of tangential traction

$$q^*(x)=-\frac{2}{\pi}x\sqrt{b^2-x^2}\left[-\int_0^b \frac{\sigma_\mathrm{b}/2\gamma}{\sqrt{b^2-y^2}(x^2-y^2)}\mathrm{d}y+\frac{4}{\pi^2}\int_0^b \frac{\int_b^a \frac{f\bar{p}}{(t^2-y^2)\sqrt{1-(t/a)^2}}t\,\mathrm{d}t}{\sqrt{b^2-y^2}(x^2-y^2)}\mathrm{d}y\right] \tag{46}$$

where the first integral is equal to zero, whereas for the second we get

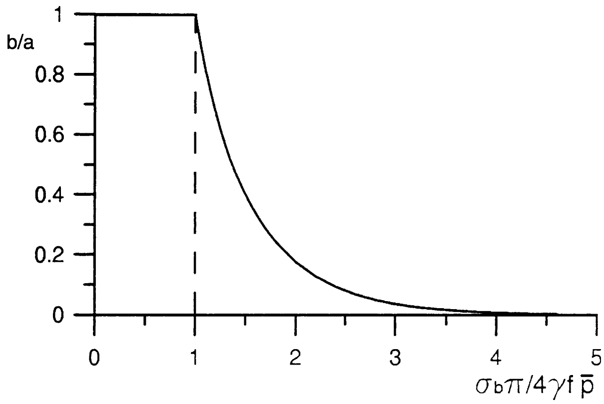

Fig. 3. Bulk load only: stick area semiwidth as a function of non-dimensional bulk load $(\pi/4)(\sigma_\mathrm{b}/\gamma f\bar{p})$.

$$\int_b^a \frac{f\bar{p}}{\sqrt{1-(t/a)^2}}t\left[\int_0^b \frac{1}{(t^2-y^2)(x^2-y^2)\sqrt{b^2-y^2}}\mathrm{d}y\right]\mathrm{d}t \tag{47}$$

Solving the integral in the square brackets we find

$$q^*(x) = \frac{4}{\pi^2}x\sqrt{b^2-x^2}\int_b^a \frac{f\bar{p}}{\sqrt{1-(t/a)^2}\sqrt{t^2-b^2}(t^2-x^2)}$$

The shear tractions in the stick area can be also expressed using $\Pi(n,m)$, the complete elliptic integral of the third type,

$$q^*(x) = \frac{4}{\pi^2}f\bar{p}\frac{x\sqrt{b^2-x^2}}{a^2-x^2}\Pi\left(\frac{a^2-b^2}{a^2-x^2}, 1-\frac{b^2}{a^2}\right) \tag{48}$$

In Fig. 4 the tangential tractions are plotted for different values of bulk load $\sigma_b/(4/\pi)\gamma f\bar{p}$. The surface stress is given by (27) and so for the case of $\sigma_b \leqslant (4/\pi)\gamma f\bar{p}$ we find

$$\sigma_{xx}(x,0) = \frac{\sigma_b}{2} + \frac{2\sigma_b}{\pi\gamma}\frac{x}{\sqrt{x^2-a^2}}\left(\arctan\sqrt{\frac{x-a}{x+a}} + \arctan\sqrt{\frac{x+a}{x-a}}\right), \quad |x| \geqslant a \tag{49}$$

$$\sigma_{xx}(x,0) = \frac{\sigma_b}{2} + p(x), \quad |x| \leqslant a \tag{50}$$

whereas for $\sigma_b \geqslant (4/\pi)\gamma f\bar{p}$

$$\sigma_{xx}(x,0) = \frac{\sigma_b}{2} + p(x) - \frac{8}{\pi^2}f\bar{p}\frac{\sqrt{x^2(x^2-b^2)}}{a^2-x^2}\Pi\left(\frac{a^2-b^2}{a^2-x^2}, 1-\frac{b^2}{a^2}\right), \quad |x| \geqslant b \tag{51}$$

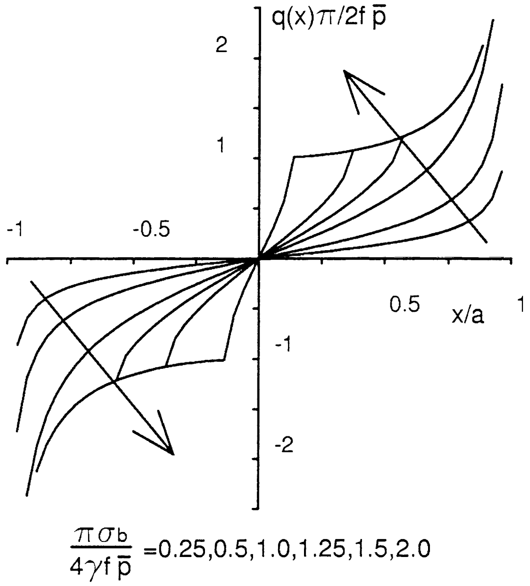

Fig. 4. Bulk load only: shear tractions for several values of bulk load $(\pi/4)(\sigma_b/\gamma f\bar{p}) = 0.25, 0.5, 1, 1.25, 1.5, 2$.

$$\sigma_{xx}(x,0) = \frac{\sigma_b}{2} + p(x), \quad |x| \leqslant b \tag{52}$$

In Fig. 4, the first three low values of bulk stresses, for which $\sigma_b/(4/\pi)\gamma f\bar{p} < 1$, show a smooth variation of shear traction, whereas the largest ones, for which $\sigma_b/(4/\pi)\gamma f\bar{p} > 1$, show a clear and abrupt change of slope in correspondence to the stick–slip interface. The surface stress state is plotted in Fig. 5, for three values of bulk stress, one lower and two higher than $(4/\pi)\gamma f\bar{p}$. Notice that for the largest values of bulk stress, the stress becomes everywhere tensile, except for a localized region near the sharp contact edges.

The K_{II} factors in such case also define the asymptotic stresses in the regions near the edges, and are found as

$$\begin{aligned} K_{II} &= \pm\frac{\sqrt{\pi a}\sigma_b}{2\gamma}, \quad \sigma_b \leqslant \frac{4}{\pi}\gamma f\bar{p} \\ K_{II} &= \pm f K_I, \quad \sigma_b \geqslant \frac{4}{\pi}\gamma f\bar{p} \end{aligned} \tag{53}$$

3.3. Tangential and bulk load

In the case of tangential and bulk load applied simultaneously, the solution of integral equation (23) is

$$q(x) = \frac{x/a}{\sqrt{1-(x/a)^2}}\frac{\sigma_b}{2\gamma} + \frac{Q}{\pi a\sqrt{1-(x/a)^2}} \tag{54}$$

in the assumption complete stick situation, when $|q(x)| \leqslant fp(x)$ in the entire contact region.

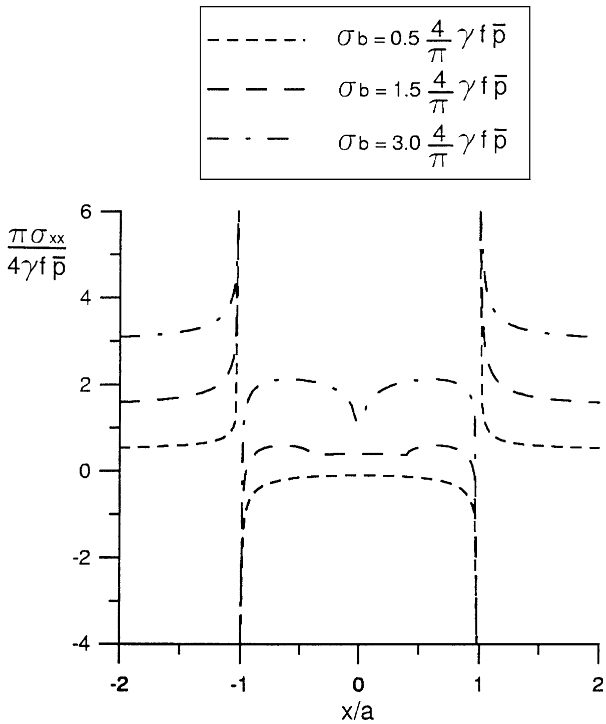

Fig. 5. Bulk load only: surface stress for several values of bulk load $(\pi/4)(\sigma_b/\gamma f\bar{p}) = 0.5, 1.5, 3$.

Accordingly, we can write that (54) is the correct solution if the following condition is satisfied

$$\frac{\sigma_{\rm b}}{2\gamma}+\frac{Q}{\pi a}\leqslant f\frac{P}{\pi a}\Rightarrow \sigma_{\rm b}\leqslant\frac{4}{\pi}\gamma f\bar{p}\left(1-\frac{Q}{fP}\right) \tag{55}$$

If the condition is not satisfied we can expect the microslip region to be next to the left edge, in opposite direction with respect to the tangential load Q. We can therefore write the tangential traction as the sum of a component of complete sliding $fp(x)$ and a corrective contribute $q^*(x)$, different from zero only in the stick area. If we indicate b as the coordinate of left edge of the stick area, we can write

$$q(x)=\begin{cases} f|p(x)|+q^*(x), & -a\leqslant x\leqslant b\\ f|p(x)|, & b\leqslant |x|\leqslant a\end{cases}$$

From the integral equation of the shear tractions (54), we get

$$\frac{1}{\pi}\int_{-a}^{b}\frac{q^*(t)}{t-x}\,{\rm d}t=\frac{\sigma_{\rm b}}{2\gamma} \tag{56}$$

After normalizing by setting $z=(t-(b-a)/2)/(b+a)/2$ and $w=(x-(b-a)/2)/(b+a)/2$ we obtain the Cauchy integral equation of the first kind in the simple form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{q^*(z)}{z-w}\,{\rm d}z=\frac{\sigma_{\rm b}}{2\gamma} \tag{57}$$

The standard solution is reported in (Hills et al., 1993) considering that the unknown function q^* can be singular for $x=-a$ but has to be non-singular in $x=-b$,

$$q^*(w)=\frac{\sigma_{\rm b}}{2\gamma\pi}\sqrt{\frac{1-w}{1+w}}\int_{-1}^{1}\frac{1+z}{\sqrt{1-z^2}(z-w)}\,{\rm d}z=-\frac{\sigma_{\rm b}}{2\gamma}\sqrt{\frac{1-w}{1+w}} \tag{58}$$

and so

$$q^*(x)=-\frac{\sigma_{\rm b}}{2\gamma}\sqrt{\frac{b-x}{a+x}}$$

The only unknown b can be calculated from the equilibrium condition

$$Q=\int_{-a}^{a}q(x)\,{\rm d}x=fP-\frac{\pi\sigma_{\rm b}}{2\gamma}\frac{b+a}{2} \tag{59}$$

and

$$b=\frac{4\gamma(fP-Q)}{\pi\sigma_{\rm b}}-a \tag{60}$$

If the (55) is not satisfied, the ratio above is less than $2a$ and so accordingly b is correctly less than a. In Fig. 6 we can observe the traction $q(x)$ plotted in non-dimensional terms as $(\pi/2)q(x)/f\bar{p}$ for $Q/fP=0.5$ and $\sigma_{\rm b}/(4/\pi)\gamma f\bar{p}=1$. Also included in the figure are the frictional limits for $(\pi/2)q(x)/f\bar{p}$, given by the pressure terms $(\pi/2)p(x)/\bar{p}$ and its value after a change of sign $-(\pi/2)p(x)/f\bar{p}$ for the reverse sliding. If the bulk load is too large there is also slip next to the right corner, i.e.

$$\frac{fP}{\pi a\sqrt{1-(x/a)^2}}-\frac{\sigma_{\rm b}}{2\gamma}\sqrt{\frac{b-x}{a+x}}\leqslant-\frac{fP}{\pi a\sqrt{1-(x/a)^2}},\quad x\to -a \tag{61}$$

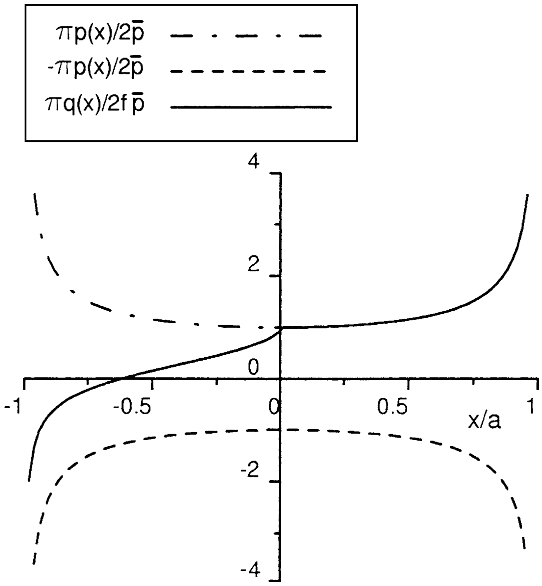

Fig. 6. Bulk and tangential loads: shear tractions for $Q/fP = 0.5$ and $(\pi/4)(\sigma_b/\gamma f\bar{p}) = 1$.

or

$$\frac{\sigma_b}{2\gamma}\sqrt{b+a} \geqslant 2\frac{fP}{\pi\sqrt{2a}} \tag{62}$$

Taking into account of (60), after some algebra we get

$$\sigma_b \geqslant \frac{4}{\pi}\gamma f\bar{p}\frac{1}{1-Q/fP} \tag{63}$$

For bulk loads larger than the above limit the only practical solution is a numerical method to find the tractions. However, since it is only the asymptotic stresses which matters, and these are known from the sliding condition at both ends, there is no need to further pursue the numerical solution. Therefore, we have three possibly situations according to the bulk load, as we can see in Fig. 7.

The K_{II} factors in the cases of $\sigma_b \leqslant (4/\pi)\gamma f\bar{p}(1-Q/fP)$ are found as

$$\begin{aligned} K_{II} &= \frac{\sqrt{\pi a}\sigma_b}{2\gamma} + \frac{Q}{\sqrt{\pi a}}, \quad x = a \\ K_{II} &= -\frac{\sqrt{\pi a}\sigma_b}{2\gamma} + \frac{Q}{\sqrt{\pi a}}, \quad x = -a \end{aligned} \tag{64}$$

whereas for $(4/\pi)\gamma f\bar{p}(1-Q/fP) \leqslant \sigma_b \leqslant (4/\pi)\gamma f\bar{p}/(1-Q/fP)$

$$\begin{aligned} K_{II} &= fK_I - \frac{\sqrt{2\pi}\sigma_b}{2\gamma}\sqrt{\frac{4\gamma(fP-Q)}{\pi\sigma_b}}, \quad x = -a \\ K_{II} &= fK_I, \quad x = a \end{aligned} \tag{65}$$

and finally for $\sigma_b \geqslant (4/\pi)\gamma f\bar{p}/(1-Q/fP)$

$$\begin{aligned} K_{II} &= -fK_I, \quad x = -a \\ K_{II} &= fK_I, \quad x = a \end{aligned} \tag{66}$$

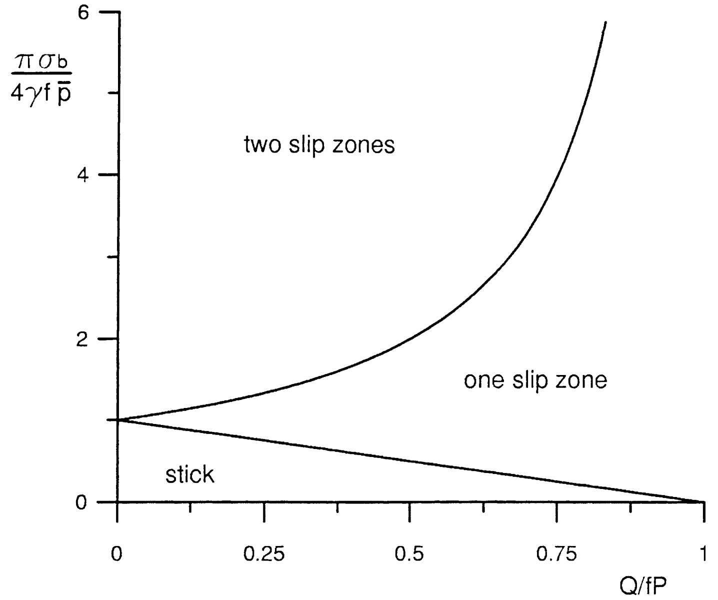

Fig. 7. Stick zone boundaries according to loading conditions.

Figs. 8 and 9 give the variation of the mode II stress intensity factor as a function of bulk and tangential loads, at the trailing and leading edge, respectively: the latter is given only for completeness (as the trailing edge value is always greater), and notice that it varies sign, for large enough bulk loads.

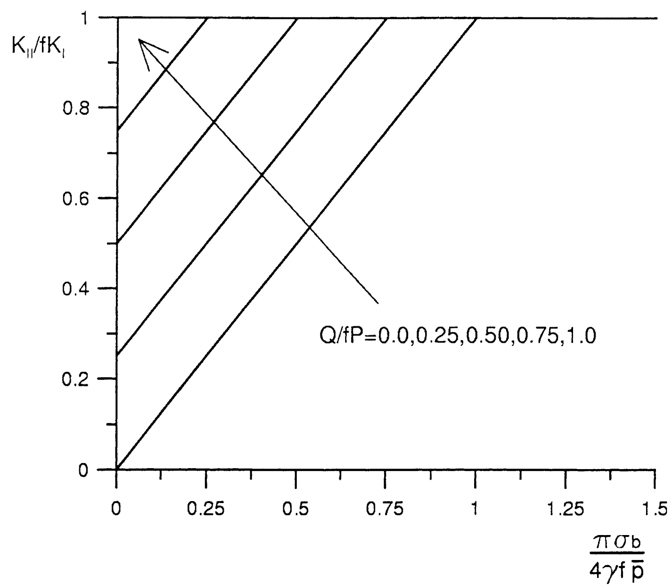

Fig. 8. Stress intensity factors ratio $K_{II}/fK_I \leqslant 1$ at the *trailing* edge, according to loading conditions.

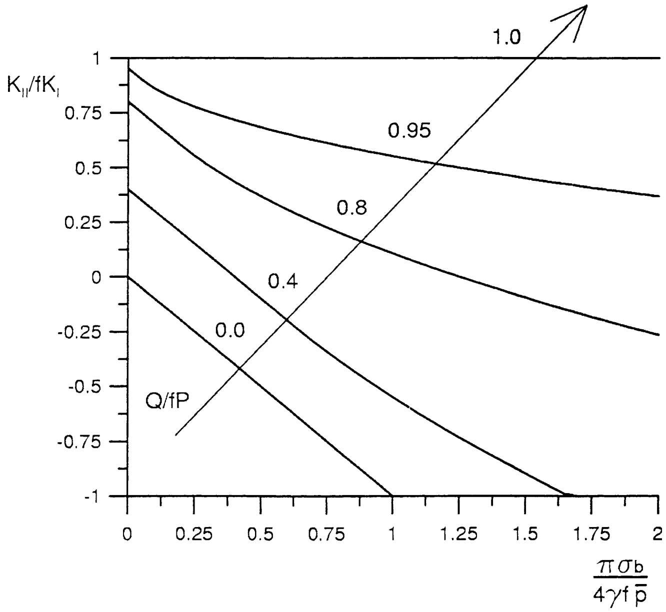

Fig. 9. Stress intensity factors ratio $K_{II}/fK_I \leqslant 1$ at the *leading* edge, according to loading conditions.

4. Consequences for the CA model

In terms of the inclination direction for the first propagation phase, we return to (14) where K_I is fixed while K_{II} depends on the load case and is an oscillating term. The CA model suggest to impose the max-

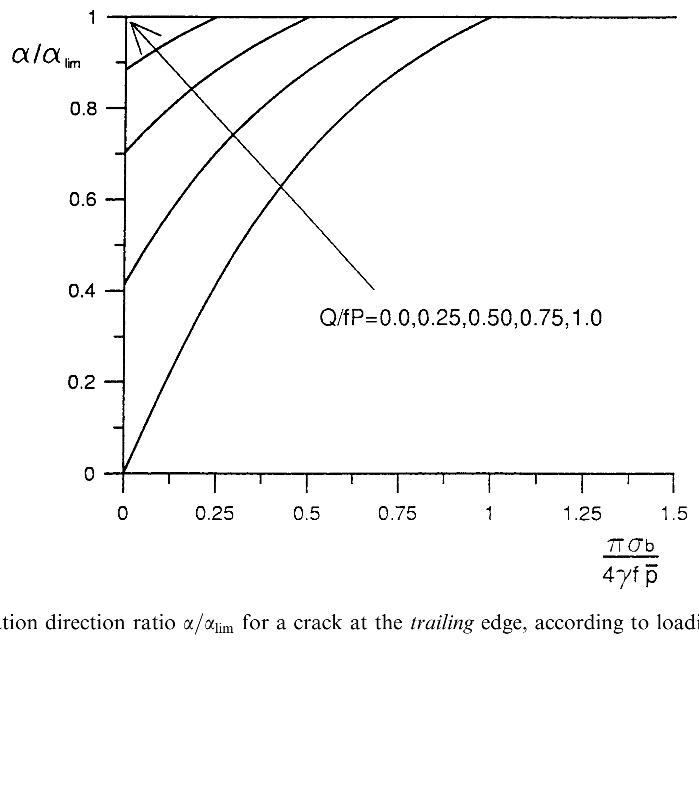

Fig. 10. Initiation direction ratio α/α_{lim} for a crack at the *trailing* edge, according to loading conditions.

imum SIF value to find the angle with the x axis, ϕ in Fig. 1b, (α is the complementary ϕ to π, i.e. $\alpha = \pi - \phi$). In the partial slip condition, clearly the ratio K_{II}/K_I is constant and is equal to the friction coefficient f, whereas it is variable in the situation of complete stick. In the latter case we have to impose k_2 to be zero. In either condition, we compute the predicted initiation angle as

$$\frac{\sin\frac{\phi}{2} + \sin\frac{3\phi}{2}}{\cos\frac{\phi}{2} + 3\cos\frac{3\phi}{2}} = -\frac{K_{II}}{K_I} \tag{67}$$

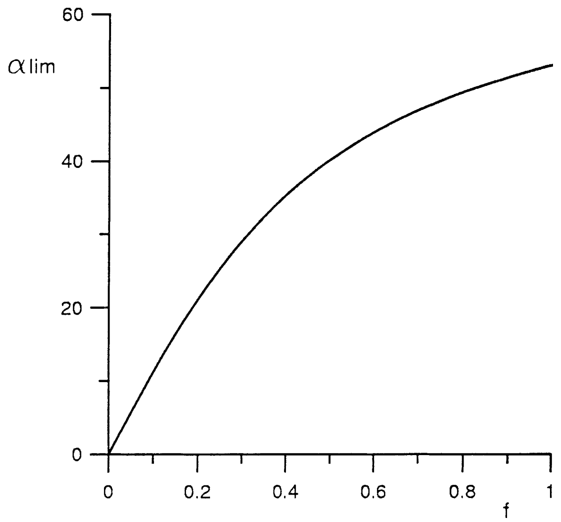

Fig. 11. Limiting initiation direction for a crack as a function to the friction coefficient f.

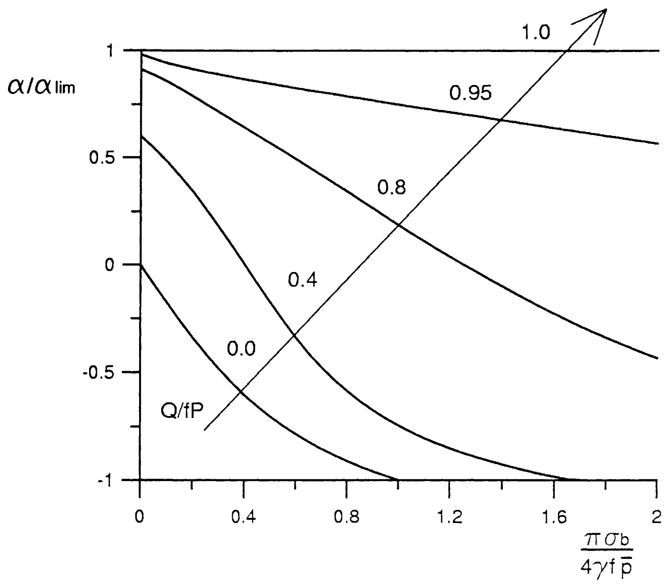

Fig. 12. Initiation direction ratio α/α_{lim} for a crack at the *leading* edge, according to loading conditions.

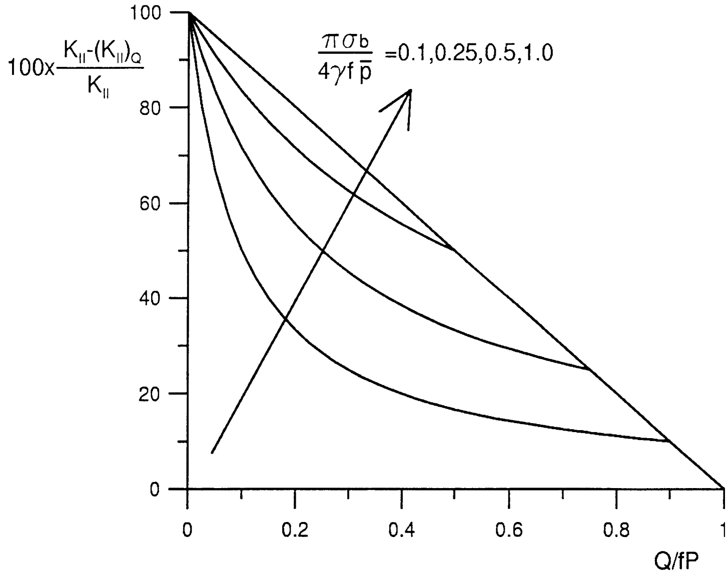

Fig. 13. Correction (in %) for the mode II stress intensity factor with respect to the original CA model, according to loading conditions.

In the case of complete stick, and $\sigma_b \leqslant (4/\pi)\gamma f\bar{p}(1 - Q/fP)$

$$\frac{\sin\frac{\phi}{2} + \sin\frac{3\phi}{2}}{\cos\frac{\phi}{2} + 3\cos\frac{3\phi}{2}} = \frac{\pi(\sigma_b)_{\max}}{4\gamma\bar{p}} + \frac{Q_{\max}}{P} \tag{68}$$

whereas for larger bulk loads, we have full slip conditions, imposing that the second term is equal to the limit value f

$$\frac{\sin\frac{\phi_{\lim}}{2} + \sin\frac{3\phi_{\lim}}{2}}{\cos\frac{\phi_{\lim}}{2} + 3\cos\frac{3\phi_{\lim}}{2}} = f \tag{69}$$

In Fig. 10 the initiation direction is shown as a ratio with $\alpha_{\lim} = \pi - \phi_{\lim}$, the limit value depending on friction coefficient f which is in turn plotted in Fig. 11. In Fig. 12 we can observe the initiation direction of crack in the leading edge (the limit angle is still obviously the same as in the trailing edge).

5. Conclusions

The contact problem for a flat punch has been solved, with a proper formulation including the bulk stress effect on the contact shear traction, improving the contact analysis of the recent CA model for FF made by Giannakopoulos et al. (1998). The corrected (higher) value is shown to occur for the mode II stress intensity factors. Specifically, for $\sigma_b \leqslant (4/\pi)\gamma f\bar{p}(1 - Q/fP)$ we find for the trailing edge (where the crack is most likely to initiate)

$$K_{II,\max} = \frac{\sqrt{\pi a}\sigma_b}{2\gamma} + \frac{Q}{\sqrt{\pi a}}, \quad x = a \tag{70}$$

whereas for larger bulk stresses,

$$K_{II,\max} = fK_I, \quad x = a \tag{71}$$

In the previous CA model K_{II} is calculated taking into account only tangential load

$$K_{II} = (K_{II})_Q = \frac{Q}{\sqrt{\pi a}} \tag{72}$$

Hence, the error in neglecting bulk stress varies as shown in Fig. 13,

$$\text{err\%} = 100 \times \frac{K_{II} - (K_{II})_Q}{K_{II}} \tag{73}$$

which is higher for low values of Q/fP and larger bulk stresses. The original CA does not consider the effect of bulk stress, and the error may become very large for bulk stresses already as low as $\sigma_b = 0.1(4/\pi)\gamma f \bar{p}$ if at the same time Q/fP is lower than, say, 0.5. Indeed the error can grow up to the limiting 100% (according to the definition given in Fig. 13) when the bulk stress only is present ($Q/fP = 0$) as in this case the original CA model would not imply any stress intensity at all. Vice versa, given that there is a limit to the value of the asymptotic tractions near the contact edges (given by friction), the maximum error for $Q/fP > 0$ is smaller than 100%, and indeed follows a linear variation with Q/fP, as

$$\text{err\%}_{\max} = 100(1 - Q/fP) \tag{74}$$

Acknowledgements

Giuseppe Macina is pleased to acknowledge the support from CEMeC—Centre of Excellence in Computational Mechanics of Politecnico di Bari.

References

Cattaneo, C., 1938. Sul contatto di due corpi elastici: distribuzione locale degli sforzi. Rendiconti dell'Accademia Nazionale dei Lincei 27, 342–348, 434–436, 474–478.

Ciavarella, M., 1998a. The generalized Cattaneo partial slip plane contact problem. I—Theory. Int. J. Solids Struct. 35 (18), 2349–2362, II-Examples, pp. 2363–2378.

Ciavarella, M., 1998b. Tangential loading of general 3D contacts. J. Appl. Mech. 64 (4), 998–1003.

Ciavarella, M., Demelio, G., Hills, D.A., 1999. An analysis of rotating bending fretting fatigue tests using 'bridge' specimens. In: Hoeppner, D.W., Chandrasekaran, V., Elliot, C.B. (Eds.), Fretting Fatigue: Current Technologies and Practices, ASTM STP 1367. American Society for Testing and Materials, West Conshohocken, PA.

Ciavarella, M., Hills, D.A., 1999. Brief note: some observation on oscillating tangential forces and wear in general plane contacts. Eur. J. Mech. A/Solids 18, 491–497.

Eden, E.M., Rose, W.N., Cunningham, F.L., 1911. The endurance of metals. Proc. Inst. Mech. Engrs. 875.

Giannakopoulos, A.E., Lindley, T.C., Suresh, S., 1998. Overview NO. 129, Aspects of equivalence between contact mechanics and fracture mechanics: theoretical connections and a life-prediction methodology for fretting-fatigue. Acta Mater. 46 (9), 2955–2968.

Giannakopoulos, A.E., Lindley, T.C., Suresh, S., 2000. Similarities of stress concentration in contact at round punches and fatigue at notches: implication to fretting fatigue crack initiation. Fatigue Fract. Eng. Mater. Struct. 23, 561–571.

Hills, D.A., Nowell, D., Sackfield, A., 1993. Mechanics of Elastic Contacts. Butterworth Heinemann, Oxford.

Jaeger, J., 1998. A new principle in contact mechanics. ASME J. Tribol. 120 (4), 677–684.

Johnson, K.L., 1985. Contact Mechanics. Cambridge University Press, Cambridge.

Mindlin, R.D., 1949. Compliance of elastic bodies in contact. J. Appl. Mech. 16, 259–268.

Mugadu, A., 2002. An asymptotic approach to crack initiation in fretting fatigue of complete contacts. J. Mech. Phys. Solids 50, 531–547.

Nishioka, K., Hirakawa, K., 1969. Fundamental investigation of fretting fatigue—part 3. Some phenomena and mechanisms of surface cracks—part 4. The effect of mean stress—part 5. The effect of relative slip amplitude. Bull. JSME 12, 397–407, 408–414, 692–697.

Nowell, D., Hills, D.A., 1987. Mechanics of fretting fatigue tests. Int. J. of Mech. Sci. 29 (5), 355–365.

Nowell, D., Hills, D.A., 1990. Crack initiation criteria in fretting fatigue. Wear 136, 329–343.

Spence, D.A., 1973. An eigenvalue problem for elastic contact with finite friction. Proc. Cambridge Philos. Soc. 73, 249–268.

Tomlinson, G.A., 1927. The rusting of steel surfaces in contact. Proc. R. Soc. London A 115, 472–483.